Solution of heterotic Killing spinor equations and special geometry

George Papadopoulos[1] and Ulf Gran[2]

[1] Department of Mathematics
King's College London
Strand
London WC2R 2LS, UK

[2] Fundamental Physics
Chalmers University of Technology
SE-412 96 Göteborg, Sweden

Abstract

We outline the solution of the Killing spinor equations of the heterotic supergravity. In addition, we describe the classification of all half supersymmetric solutions.

1 Introduction

Supersymmetric supergravity backgrounds are solutions of the field equations of supergravity theories which in addition solve a set of first order equations, the Killing spinor equations. These solutions are triplets $(\mathcal{M}, g, F)$, where $\mathcal{M}$ is a Lorentzian manifold with metric g, and F are the fluxes of supergravity theories which is a collection of forms on $\mathcal{M}$. The field equations of supergravity theories consist of the Einstein equation as well as appropriate Maxwell type of equations for F. The Killing spinor equations are determined from the supersymmetry transformations of the fermions[1] of the supergravity theories. Moreover their integrability conditions imply some of the supergravity field equations.

Recently, there is much interest in systematically understanding the supersymmetric solutions of the supergravity theories. This has been mostly motivated by the applications that these solutions have in string theory, M-theory and in the AdS/CFT correspondence. Apart from this, the supersymmetric supergravity solutions are the gravitational analogues of gauge theory solitons and instantons, and so their classification is interesting in its own right.

The main aim of this article is to outline the classification of the solutions of the Killing spinor equations of the heterotic supergravity [1, 2, 3]. Moreover, all supersymmetric solutions which preserve 8 Killing spinors will be described [4]. This material is partly based on work done in collaboration with Diederik Roest, Philipp Lohrmann and Peter Sloane as well as on material published by one of the authors. In addition, this paper contains a refinement of the results of the first two papers. In particular, a more concise description of the geometry of the backgrounds with non-compact holonomy is given in terms of certain Clifford algebras of endomorphisms.

This paper is organized as follows. In section two, the Killing spinor and field equations of the heterotic supergravity are given. We also summarize the main ingredients of the method that we use to solve the Killing spinor equations. In section three, the gravitino Killing spinor equation is solved. In section four, an outline of the solution of the dilatino Killing spinor equation is given. In section five, the geometry of supersymmetric backgrounds with non-compact and compact holonomy is described. In section six, we solve the field equations of the heterotic supergravity for all backgrounds preserving 8 supersymmetries, i.e. we describe all half supersymmetric backgrounds.

2 Killing spinor and field equations

2.1 Killing spinor and field equations

The spacetime is a 10-dimensional Lorentzian manifold $\mathcal{M}$. The bosonic fields of heterotic supergravity are a metric g, a 3-form field strength H, the dilaton scalar Φ, and a gauge connection A with curvature $F = dA - A \wedge A$. The gauge group of A is either $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$. Though this restriction on the gauge group does not affect

---

[1]After considering the supersymmetry transformations, in what follows all the fermionic fields are set to zero.

most of the analysis that will follow.

The gravitino, gaugino and dilatino Killing spinor equations of the heterotic supergravity are

$$
\begin{aligned}
\mathcal{D}_M \epsilon &\equiv \hat{\nabla}_M \epsilon + \mathcal{O}(\alpha'^2) = 0 \ , \quad \mathcal{A}\epsilon \equiv (\Gamma^M \partial_M \Phi - \frac{1}{12} H_{MNL} \Gamma^{MNL})\epsilon + \mathcal{O}(\alpha'^2) = 0 \ , \\
\mathcal{F}\epsilon &\equiv F_{MN} \Gamma^{MN} \epsilon + \mathcal{O}(\alpha'^2) = 0 \ ,
\end{aligned}
\tag{1}
$$

respectively, where ϵ is a real positive chirality spinor (Majorana-Weyl) of $Spin(9,1)$ and $\hat{\nabla} = \nabla + \frac{1}{2}H$ is a metric connection with torsion H. Moreover, $\{\Gamma_M\}$ is a basis of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{9,1})$,

$$
\Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2g_{MN} \ ,
\tag{2}
$$

and $M, N, L = 0, \dots, 9$. More details about the notation can be found in [1, 2]. The Killing spinor equations have been expressed as an expansion in the parameter α'. They are known to the order indicated but it is expected that they receive corrections to higher orders.

The 3-form field strength H is not closed but is modified at order α' because of the Green-Schwarz anomaly cancelation mechanism as

$$
dH = -\frac{\alpha'}{4}\left(\mathrm{tr}\ \check{R}^2 - \mathrm{tr}\ F^2\right) + \mathcal{O}(\alpha'^2) \ ,
\tag{3}
$$

where $\check{R}$ is the curvature of $\check{\nabla} = \nabla - \frac{1}{2}H$.

The field equations (in the string frame) to lowest order in α' are

$$
\begin{aligned}
E_{MN} &\equiv R_{MN} + \frac{1}{4} H^R{}_{ML} H^L{}_{NR} + 2\nabla_M \partial_N \Phi \\
&\qquad + \frac{\alpha'}{4}[\check{R}_{ML,QR}\check{R}_N{}^{L,QR} - F_{MLab}F_N{}^{Lab}] + \mathcal{O}(\alpha'^2) = 0 \ , \\
LH_{PR} &\equiv \nabla_M[e^{-2\Phi}H^M{}_{PR}] + \mathcal{O}(\alpha'^2) = 0 \ , \\
LF_M &\equiv \hat{\nabla}^M[e^{-2\Phi}F_{MN}] + \mathcal{O}(\alpha'^2) = 0 \ .
\end{aligned}
\tag{4}
$$

The linear term in α' in the Einstein equation, which arises from the 2-loop sigma model beta function calculation [5], is necessary for consistency with (3), see e.g. [6]. The remaining two field equations are Maxwell type of equations for the 3-form flux H and the 2-form gauge field strength F. The field equation for the dilaton is implied from those above up to a constant.

2.2   Method

The method we shall use to solve the Killing spinor equations of the heterotic supergravity is *spinorial geometry* [7]. It is based on

- the gauge symmetry of Killing spinor equations,

- a description of spinors in term of forms,

2

- a harmonic oscillator basis in the space of spinors.

The basic strategy is to use the gauge symmetry of the Killing spinor equations to choose a canonical form for the Killing spinors or their normals. Then writing the Killing spinors in terms of forms, these can be substituted into the Killing spinor equations. The resulting expressions are solved by utilizing the linearity of the Killing spinor equations and expanding them in the harmonic oscillator basis in the space of spinors.

The above method is very effective particularly for the solutions of the Killing spinor equations with small or near maximal number of supersymmetries[2]. It can be implemented equally efficiently in analytic or computer calculations.

Returning to the Killing spinor equations of heterotic supergravity, it is convenient to solve them in the order

$$\text{gravitino} \rightarrow \text{gaugino} \rightarrow \text{dilatino} .$$

The solution of the gaugino Killing spinor equation has been given in [3], and it is similar to that of the gravitino. Because of this in the analysis that follows, we shall focus on the solution of the gravitino and dilatino Killing spinor equations [1, 2].

To apply the spinorial geometry method to the heterotic supergravity, first observe that the gauge symmetry of the Killing spinor equations (1) is $Spin(9,1)$. This coincides with the holonomy group of $\hat{\nabla}$, $\mathrm{hol}(\hat{\nabla})$, for generic backgrounds. This equality is the main reason that all the solutions of the Killing spinor equations of the heterotic supergravity can be found.

2.3   Spinors

One of the ingredients of spinorial geometry is the description of spinors in terms of forms. This is a well-known realization of the spinor representations, see e.g. [9], and it has been used in [10] to give explicitly the parallel spinors of Riemannian manifolds with special holonomy. This description of spinors can be extended to the Lorentzian case. For later use, we give the form realization of spinor representations[3] of $Spin(9,1)$, see also [1].

Consider $\mathbb{C}^5 = \mathbb{C} < e^1, \ldots, e^5 >$, where $e^1, \ldots, e^5$ is a Hermitian basis with respect to the $< \cdot, \cdot >$ inner product. The space of Dirac spinors of $Spin(9,1)$ is $\Delta_c = \Lambda^*(\mathbb{C}^5)$. The basis $\{\Gamma_A\}$ of Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{9,1})$ acts on Δ_c as

$$\begin{aligned}
\Gamma_0 \psi &= -e_5 \wedge \psi + e_5 \lrcorner \psi , & \Gamma_5 \psi &= e_5 \wedge \psi + e_5 \lrcorner \psi , \\
\Gamma_i \psi &= e_i \wedge \psi + e_i \lrcorner \psi , & \Gamma_{5+i} \psi &= i e_i \wedge \psi - i e_i \lrcorner \psi , & i = 1, \ldots, 4,
\end{aligned} \qquad (5)$$

where $\psi \in \Delta_c$ and $\lrcorner$ is the adjoint operation of $\wedge$ with respect to $< \cdot, \cdot >$. It is easy to verify that $\{\Gamma_A\}$ satisfies the Clifford algebra relation $\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}$, where η is the Minkowski metric. Δ_c is a reducible $Spin(9,1)$ representation and decomposes into

---

[2]For the classification of near maximally supersymmetric backgrounds using spinorial geometry see [8].

[3]The spin groups considered here are the double covers of the component of the Lorentz group connected to the identity.

3

two complex chiral representations $\Delta_c^+ = \Lambda^{\mathrm{even}}(\mathbb{C}^5)$ and $\Delta_c^- = \Lambda^{\mathrm{odd}}(\mathbb{C}^5)$. These are the complex Weyl representations of $Spin(9,1)$.

It is well-known that $Spin(9,1)$ admits two inequivalent *real* chiral representations, the Majorana-Weyl representations. These are constructed by imposing a reality condition on $\Delta_c^\pm$. This is achieved by using the reality map $R = \Gamma_{6789}*$ which is anti-linear, $R^2 = 1$, and commutes in the action of $Spin(9,1)$. So the real spinors satisfy

$$\eta^* = \Gamma_{6789}\eta \ . \tag{6}$$

For example the real and imaginary components of the complex spinor 1 are $1 + e_{1234}$ and $i(1 - e_{1234})$, respectively, where $e_{1234} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We denote the real subspaces of $\Delta_c^\pm$ with $\Delta_{16}^\pm$.

The spacetime form bilinears associated with the spinors ψ, θ. are given as

$$\alpha(\psi, \theta) \equiv \frac{1}{k!} B(\psi, \Gamma_{A_1 \ldots A_k} \theta) \ e^{A_1} \wedge \cdots \wedge e^{A_k} \ , \qquad k = 0, \ldots, 9 \ . \tag{7}$$

where

$$B(\psi, \theta) = < \mathrm{B}(\psi^*), \theta > \ , \tag{8}$$

is the $Spin(9,1)$-invariant Majorana bilinear inner product on Δ_c and the linear map B is $\mathrm{B} = \Gamma_{06789}$.

3 Gravitino Killing spinor equation

Let us assume that the spacetime $\mathcal{M}$ is simply connected. To investigate the solutions of the gravitino Killing spinor equation [1, 2], consider the integrability condition

$$\hat{R}\epsilon = 0 \ . \tag{9}$$

This equation has solutions if either $\hat{R} = 0$, and so $\mathcal{M}$ is parallelisable, or the solutions $(\epsilon_1, \ldots, \epsilon_L)$ have a non-trivial isotropy group $\mathrm{Stab}(\epsilon_1, \ldots, \epsilon_L) \subset Spin(9,1)$ and

$$\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{Stab}(\epsilon_1, \ldots, \epsilon_L) \ . \tag{10}$$

In the former case, $\mathcal{M}$ is either a Lorentzian manifold or a product of a Lorentzian group manifold with S^7 [11]. If in addition, one assumes that $dH = 0$, then $\mathcal{M}$ is a Lorentzian group manifold. The Lorentzian groups manifolds have been classified in [12]. Locally up to dimension ten, they are products of the Lorentzian groups $\mathbb{R}$, $SL(2, \mathbb{R})$, CW_{2k} with the Riemannian group manifolds $U(1)$, $SU(2)$ and $SU(3)$, where CW_{2k} are the group manifolds[4] associated with the Cahen-Wallace spaces.

In the latter case, one can determine the subgroups of $Spin(9,1)$ which are isotropy groups of spinors. These have been tabulated in table 1. This table has been constructed in stages [13, 1, 2]. In the same table, a basis in the space of parallel spinors is given for each case. These bases have been written down explicitly using the form notation

---

[4]These are plane waves with wave profile given by the square of a skew-symmetric matrix.

for spinors explained in section 2.3 and they are determined up to $Spin(9,1)$ gauge transformations.

A straightforward observation of results tabulated in table 1 reveals that there are two types of isotropy groups of spinors that occur distinguished by their topology, the compact and non-compact ones. The non-compact isotropy groups are of the type $K \times \mathbb{R}^8$, where K is compact. As we shall explain this distinction is useful in the description of geometry of the associated spacetimes. Most of the isotropy groups that occur are of the Berger type. However there are some exceptions which do not appear in the Berger list. These are whenever $\text{Stab}(\epsilon_1, \ldots, \epsilon_L)$ is $\times^2 SU(2) \ltimes \mathbb{R}^8$, $SU(2) \ltimes \mathbb{R}^8$, $U(1) \ltimes \mathbb{R}^8$ and $\mathbb{R}^8$.

L	$\text{Stab}(\epsilon_1, \ldots, \epsilon_L)$	$\Sigma(\mathcal{P})$	$\epsilon_1, \ldots, \epsilon_L$
1	$Spin(7) \ltimes \mathbb{R}^8$	$Spin(1,1)$	$1 + e_{1234}$
2	$SU(4) \ltimes \mathbb{R}^8$	$Spin(1,1) \times U(1)$	1
3	$Sp(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(2)$	$1, \quad i(e_{12} + e_{34})$
4	$\times^2 SU(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times (\times^2 Sp(1))$	$1, \quad e_{12}$
5	$SU(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(2)$	$1, \quad e_{12}, \quad e_{13} + e_{24}$
6	$U(1) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(4)$	$1, \quad e_{12}, \quad e_{13}$
8	$\mathbb{R}^8$	$Spin(1,1) \times Spin(8)$	$1, e_{ij}, i, j \leq 4$
2	G_2	$Spin(2,1)$	$1 + e_{1234}, e_{15} + e_{2345}$
4	$SU(3)$	$Spin(3,1) \times U(1)$	$1, e_{15}$
8	$SU(2)$	$Spin(5,1) \times SU(2)$	$1, e_{12}, e_{15}, e_{25}$
16	$\{1\}$	$Spin(9,1)$	$1, e_{ij}, e_{i5}$

Table 1: In the columns are the numbers of parallel spinors, their isotropy groups and their $\Sigma(\mathcal{P})$ groups, respectively. The $\Sigma(\mathcal{P})$ groups are a product of a $Spin$ group and an R-symmetry group of a lower-dimensional supergravity theory.

4 Dilatino Killing spinor equation

Suppose that we have a solution of the gravitino Killing spinor equation and the $\hat{\nabla}$-parallel spinors span an L-plane $\mathcal{P}_L$. Typically only some of the $\hat{\nabla}$-parallel spinors will be Killing, i.e. they will solve both the gravitino and dilatino Killing spinor equations. Following [2] to solve the dilatino Killing spinor equation, one has to choose representatives for the Killing spinors up to $Spin(9,1)$ gauge transformations. It turns out that given the $\hat{\nabla}$-parallel spinors, a suitable choice of gauge transformations is

$$\Sigma(\mathcal{P}_L) = \text{Stab}(\mathcal{P}_L)/\text{Stab}(\epsilon_1, \ldots, \epsilon_L) , \tag{11}$$

where $\text{Stab}(\mathcal{P}_L) = \{\ell \in Spin(9,1)| \quad \ell \mathcal{P}_L \subseteq \mathcal{P}_L\}$. The quotient with $\text{Stab}(\epsilon_1, \ldots, \epsilon_L)$ is taken because this subgroup acts with the identity on $\mathcal{P}_L$. The $\Sigma(\mathcal{P}_L)$ groups have been tabulated in table 1.

The analysis of the solutions of the dilatino Killing spinor equation proceeds as follows. Given a solution of the gravitino Killing spinor equations, one determines $\mathcal{P}_L$. Now

L	$\mathrm{Stab}(\epsilon_1,\ldots,\epsilon_L)$	N
1	$Spin(7) \ltimes \mathbb{R}^8$	1(1)
2	$SU(4) \ltimes \mathbb{R}^8$	1(1), 2(1)
3	$Sp(2) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1)
4	$(\times^2 SU(2)) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1), 4(1)
5	$SU(2) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1), 4(1), 5(1)
6	$U(1) \ltimes \mathbb{R}^8$	1(1), 2(1), 3(1), 4(1), 5(1), 6(1)
8	$\mathbb{R}^8$	1(1), 2(1), 3(1), 4(1), 5(1), 6(1), 7(1), 8(1)
2	G_2	1(1), 2(1)
4	$SU(3)$	1(1), 2(2), 3(1), 4(1)
8	$SU(2)$	1(1), 2(2), 3(3), 4(6), 5(3), 6(2), 7(1), 8(1)
16	$\{1\}$	8(2), 10(1), 12(1), 14(1), 16(1)

Table 2: The number in parenthesis indicates the multiplicity of the different solutions that occur for the same number of Killing spinors.

suppose only one of the parallel spinors is Killing. This can be chosen up to $\Sigma(\mathcal{P}_L)$ gauge transformations. Therefore the distinct solutions of the dilatino Killing spinor equation are labeled by the different type of orbits, $\mathcal{O}_{\Sigma(\mathcal{P}_L)}(\mathcal{P}_L)$ of $\Sigma(\mathcal{P}_L)$ in $\mathcal{P}_L$.

Having established a procedure to choose the first Killing spinor, one can proceed inductively. Let $\mathcal{K}_N$ denote the subspace of $\mathcal{P}_L$ spanned by the first N Killing spinors, $N < L$. One writes

$$0 \to \mathcal{K}_N \to \mathcal{P}_L \to \mathcal{P}_L/\mathcal{K}_N \to 0 \ . \tag{12}$$

The task is to determine $\mathcal{K}_{N+1}$. For this one has to choose an additional Killing spinor $\epsilon_{N+1} \in \mathcal{P}_L$ which is linearly independent from those in $\mathcal{K}_N$. For this, one uses as a gauge group

$$\mathrm{Stab}(\mathcal{K}_N) = \{\ell \in \Sigma(\mathcal{P}_L)| \quad \ell \mathcal{K}_N \subseteq \mathcal{K}_N\} \ . \tag{13}$$

Because of the linearity of the dilatino Killing spinor equations, one can view the additional Killing spinor ϵ_{N+1} as element of $\mathcal{P}_L/\mathcal{K}_N$. Thus, the distinct choices of ϵ_{N+1} are labeled by the different type of orbits, $\mathcal{O}_{\mathrm{Stab}(\mathcal{K}_N)}(\mathcal{P}_L/\mathcal{K}_N)$, of $\mathrm{Stab}(\mathcal{K}_N)$ in $\mathcal{P}_L/\mathcal{K}_N$. In practise this procedure is carried out for $N \leq L/2$. If $N > L/2$, then a similar procedure can be devised for selecting the normals to the Killing spinors in $\mathcal{P}_L$. Using the above procedure, the dilatino Killing spinor equation has been solved in all cases and the possibilities that arise have been tabulated in table 2.

It is clear from table 2 that the backgrounds for which $\mathrm{hol}(\hat{\nabla}) \subseteq K \ltimes \mathbb{R}^8$ can be characterized by the number L of parallel spinors, and the number N of Killing spinors. This is because each case that appears has multiplicity one. This is not the case for backgrounds for which $\mathrm{hol}(\hat{\nabla})$ is compact. For these some information about the embedding of $\mathcal{K}_N$ in $\mathcal{P}_L$ is necessary to characterize the geometry.

Another result that becomes evident from table 2 is that, apart from the case with $\mathrm{Stab}(\epsilon_1,\ldots,\epsilon_L) = \{1\}$, for any given L, the Killing spinor equations have solutions for

any $1 \leq N \leq L$. This is a consequence of the dilatino Killing spinor equation. However not all cases are independent. For example, given $\mathcal{P}_L$, it is clear that all backgrounds with $N < L$ have the same $\hat{\nabla}$-parallel spinors. Therefore, one expects that the geometry of all these backgrounds, called descendants in [2], must have some similarity with that of backgrounds with $N = L$. This indeed is the case and it has been shown in [2] that a relation can be established using the field equations of the theory. This relates the backgrounds lying horizontally in table 2.

There is also a relation between the geometries of backgrounds lying diagonally in table 2. This will be described separately for the compact and non-compact cases below.

5 Spacetime geometry

5.1 Non-compact holonomy

In table 2, there are 29 different types of supersymmetric backgrounds for which $\mathrm{hol}(\hat{\nabla})$ is *non-compact*. However it is not necessary to investigate them separately because some of them are special cases of others. This follows from the results of [2], where all the Killing spinors are stated explicitly. To outline this relation consider the case of (L, N) background, $N \neq 7$, i.e. a background with L parallel and N Killing spinors, $N < L$. As has already been mentioned the pair (L, N) uniquely determines the background. It turns out that the Killing spinors of this background are *identical* to those of (N, N) background. Thus the geometry of the (L, N) backgrounds is a special case of that of (N, N) backgrounds, $N \neq 7$. This establishes a relation between the geometries of backgrounds lying diagonally in table 2. Therefore, it suffices to investigate the geometry of backgrounds for which all parallel spinors are Killing, i.e. only that of the (L, L) backgrounds for $L = 1, 2, 3, 4, 5, 6, 8$. The $(8, 7)$ backgrounds are special and should be treated separately. This has been done in [2] and we shall not expand on this here.

5.1.1 Geometry

The spacetime of (L, L) backgrounds admits $\hat{\nabla}$-parallel or fundamental forms (7) constructed from Killing spinor bilinears. It turns out that the fundamental forms of backgrounds with $\mathrm{hol}(\hat{\nabla}) = K \ltimes \mathbb{R}^8$ are

$$e^- \ , \quad e^- \wedge \tau \ , \tag{14}$$

where e^- is a null 1-form and τ is a fundamental form of K.

The solution of the Killing spinor equations

- expresses the 3-form H in terms of the metric and fundamental forms (14), and

- imposes restrictions on the geometry of spacetime which can be written as conditions on the metric and (14).

To describe both types of conditions in detail, it is convenient to define the directions "transverse" to the lightcone. For this define the vector field e_+ using $e^-(\cdot) = g(e_+, \cdot)$.

Since e_+ is also $\hat{\nabla}$-parallel, it spans a trivial bundle I in $T\mathcal{M}$. Moreover, one has

$$0 \to I \to \text{Ker}\, e^- \to \xi_{T\mathcal{M}} \to 0 \ , \tag{15}$$

where $\text{Ker}\, e^-$ is spanned by the vector fields X of $\mathcal{M}$ annihilated by e^-, $e^-(X) = 0$. It is clear that $\xi_{T\mathcal{M}}$ has rank 8 and it is identified with the directions transverse to the lightcone.

In practise this means that one can adapt a local frame (e^-, e^+, e^i), $i = 1, \ldots, 8$, where e^+, e^i are defined up to shifts along e^-, such that the solution of the Killing spinor equations can be written as

$$
\begin{aligned}
ds^2 &= 2e^- e^+ + \delta_{ij}\, e^i e^j \ , \\
H &= e^+ \wedge de^- + \frac{1}{2}(h^{\mathfrak{k}} + h^{\mathfrak{k}^\perp})_{ij}\, e^- \wedge e^i \wedge e^j + \tilde{H} \ ,
\end{aligned}
\tag{16}
$$

where

$$\tilde{H} = \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k \ . \tag{17}$$

We have already expressed some of the components of H in terms of fundamental forms because these are universal. To identify the rest of the components, first observe that the Lie algebra $\mathfrak{k}$ of K is a subspace of $\Lambda^2(\mathbb{R}^8)$, $\mathfrak{k} \subset \Lambda^2(\mathbb{R}^8)$. So one can use the metric to write $\Lambda^2(\mathbb{R}^8) = \mathfrak{k} \oplus \mathfrak{k}^\perp$. So $h^{\mathfrak{k}}$ and $h^{\mathfrak{k}^\perp}$ are the components of the 2-form h along $\mathfrak{k}$ and $\mathfrak{k}^\perp$, respectively. $\tilde{H}$ are the components of H along the directions transverse to the lightcone. From now on forms denoted by tilde have components only along the directions transverse to the light-cone.

The Killing spinor equations determine all components of H apart from $h^{\mathfrak{k}}$. In particular, $h^{\mathfrak{k}^\perp}$ and $\tilde{H}$ are determined in terms of the metric and the fundamental forms. However these expressions are case dependent. We shall mostly focus on $\tilde{H}$. The expression for $h^{\mathfrak{k}^\perp}$ can be found in [1, 2].

Next observe that

$$\hat{\nabla} e^- = 0 \iff e_+ \text{ Killing} \ , \quad de^- = e_+ H \ . \tag{18}$$

So $\mathcal{M}$ admits a single null Killing vector field. This condition on the geometry is universal. There are additional conditions which are case dependent. We shall mention these in the appropriate section.

5.1.2 $Spin(7) \ltimes \mathbb{R}^8$

Let $\phi = \tilde{\phi}$ be the self-dual fundamental 4-form of $Spin(7)$. In addition to the conditions that are universal and mentioned already, the Killing spinor equations imply that

$$\tilde{H} = -\star \tilde{d}\phi + \star(\tilde{\theta}_\phi \wedge \phi) \ , \tag{19}$$

and

$$\partial_+ \Phi = 0 \ , \quad de^- \in \mathfrak{spin}(7) \oplus_s \mathbb{R}^8 \ ,$$

8

$$2\partial_i \Phi - (\tilde{\theta}_\phi)_i - H_{-+i} = 0 \ , \tag{20}$$

where

$$\tilde{\theta}_\phi = -\frac{1}{6} \star (\star \tilde{d}\phi \wedge \phi) \tag{21}$$

is a Lee form, $\star$ is the Hodge duality operation along the transverse directions, and $\tilde{d}$ is the exterior derivative again evaluated along the transverse directions. It is clear that $\tilde{H}$ can be expressed in terms of the fundamental from ϕ. The expression is similar to that for 8-manifolds with a $Spin(7)$ structure and compatible $Spin(7)$ connection with skew-symmetric torsion [14]. The dilaton is invariant under the action of the vector field e_+. The second condition in (20) is a geometric condition which restricts the twist of the vector field e_+. In turn it implies that $e^- \wedge \phi$ is invariant under the action of e_+. The last condition can also be perceived as a geometric condition which expresses the Lee form $\tilde{\theta}_\phi$ in terms of the dilaton.

5.1.3 $SU(4) \ltimes \mathbb{R}^8$

Let $\omega_I = \tilde{\omega}_I$ and $\chi = \tilde{\chi}$ be the Hermitian and the (4,0) fundamental forms of $SU(4)$, respectively. I is an almost complex structure in $\xi_{T\mathcal{M}}$ associated with ω_I. The Killing spinor equations imply that

$$\tilde{H} = -i_{\bar{I}}d\omega_I = - \star (\tilde{d}\omega_I \wedge \omega_I) - \frac{1}{2} \star (\tilde{\theta}_{\omega_I} \wedge \omega_I \wedge \omega_I) \ , \tag{22}$$

and

$$\begin{aligned}
\partial_+ \Phi = 0 \ , \quad & de^- \in \mathfrak{su}(4) \oplus_s \mathbb{R}^8 \ , \\
\tilde{\mathcal{N}}(I) = 0 \ , \quad & \tilde{\theta}_{\omega_I} = \tilde{\theta}_{\mathrm{Re}\,\chi} \ , \\
2\partial_i \Phi - (\tilde{\theta}_{\omega_I})_i - H_{-+i} = 0 \ ,
\end{aligned} \tag{23}$$

where $\tilde{\mathcal{N}}$ is the Nijenhuis tensor of I restricted along the transverse directions and

$$\tilde{\theta}_{\omega_I} = - \star (\star \tilde{d}\omega_I \wedge \omega_I) \ , \quad \tilde{\theta}_{\mathrm{Re}\,\chi} = -\frac{1}{4} \star (\star \tilde{d}\mathrm{Re}\,\chi \wedge \mathrm{Re}\,\chi) \ , \tag{24}$$

are the Lee forms of ω_I and $\mathrm{Re}\,\chi$, respectively. The expression for $\tilde{H}$ is as that for the skew-symmetric torsion of the Bismut connection for 2n-manifolds with a $U(n)$ structure, see also [15]-[23].

There are two new type of conditions that appear in (23) compared to those which we have analyzed for the $Spin(7) \ltimes \mathbb{R}^8$ case. The first is the vanishing of the Nijenhuis tensor for I. This is a consequence of the dilatino Killing spinor equation. The other is the equality between the Lee forms $\tilde{\theta}_{\omega_I}$ and $\tilde{\theta}_{\mathrm{Re}\,\chi}$. This is required for the existence of a compatible connection with skew-symmetric torsion on 8-dimensional manifolds with an $SU(4)$ structure.

N	$\mathrm{Stab}(\epsilon_1,\dots,\epsilon_L)$	Clifford
2	$SU(4) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R})$
3	$Sp(2) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^2)$
4	$(\times^2 SU(2)) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^3)$
5	$SU(2) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^4)$
6	$U(1) \ltimes \mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^5)$
7	$\mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^6)$
8	$\mathbb{R}^8$	$\mathrm{Cliff}(\mathbb{R}^7)$

Table 3: The number of Killing spinors is given in the first column. In the second column the isotropy group of the parallel spinors is given. In the last column the associated Clifford algebra of endomorphisms is indicated.

5.1.4 $\quad Sp(2) \ltimes \mathbb{R}^8$, $\times^2 SU(2) \ltimes \mathbb{R}^8$, $SU(2) \ltimes \mathbb{R}^8$ and $U(1) \times \mathbb{R}^8$

The fundamental forms of all these backgrounds are

$$e^- \,, \quad e^- \wedge \omega_r \,, \tag{25}$$

where $\omega_r = \tilde{\omega}_r$ are Hermitian forms on the space $\xi_{T\mathcal{M}}$ transverse to the light-cone. These can be thought of as the fundamental forms of the maximal compact subgroup K in $\mathrm{Stab}(\epsilon_1,\dots,\epsilon_L) \equiv K \times \mathbb{R}^8$. These Hermitian forms ω_r and their associated endomorphisms I_r have been explicitly given in [2]. The data provided can be re-organized more efficiently in terms of Clifford algebras. In particular using the results of [2], one can show that the typical fibre of $\xi_{T\mathcal{M}}$ is an appropriate Clifford module as indicated in table 3.

To see how the fundamental Hermitian forms can be identified from table 3, first consider the $SU(4)$ case. It is clear that the almost complex structure I can be thought of as the basis element of $\mathrm{Cliff}(\mathbb{R})$. Similarly, it is known that the fundamental forms of $Sp(2)$ are Hermitian forms ω_r associated with an almost hyper-complex structure I_r. Two of the almost complex structures, say I_1 and I_2, can be identified with the two basis elements of $\mathrm{Cliff}(\mathbb{R}^2)$. The third I_3 is the product of the other two, $I_3 = I_1 I_2$, and so it is represented by the even element of $\mathrm{Cliff}(\mathbb{R}^2)$ which again is the product of the two basis elements. This construction is easily extended to all other cases. Note in particular that the geometry of both the $(8,7)$ and $(8,8)$ backgrounds can be described in this way.

Now $\tilde{H}$ can be given as in (22) with respect to any of the endomorphisms I_r, say $I = I_1$. The rest of the conditions implied from the Killing spinor equations are

$$\begin{aligned}
&\partial_+ \Phi = 0 \,, \quad i_{I_r} d\omega_r = i_{I_s} d\omega_s \,, \quad r \neq s \\
&de^- \in \mathfrak{k} \oplus_s \mathbb{R}^8 \,, \quad \tilde{\mathcal{N}}(I_r) = 0 \,, \quad \tilde{\theta}_r = \tilde{\theta}_s \,, \quad r \neq s \\
&2\partial_i \Phi - (\tilde{\theta}_r)_i - H_{-+i} = 0 \,,
\end{aligned} \tag{26}$$

where $\tilde{\theta}_r$ is the Lee form of ω_r given as in (24). The above conditions can be easily derived from those of the Killing spinor equations for the $SU(4) \ltimes \mathbb{R}^8$ case. This can be done by requiring that the conditions that are valid for the I endomorphism should now be valid for all I_r endomorphisms.

10

5.1.5 $\mathbb{R}^8$

It remains to state the conditions for the (8,8) case. It turns out that the Killing spinor equations imply that

$$e^- \wedge de^- = 0 \ , \quad \tilde{H} = 0 \ . \tag{27}$$

In section 6.2, we shall classify all such backgrounds by solving both the above conditions and associated field equations.

5.2 Compact holonomy

Table 2 indicates that there are 32 cases that we should consider. So it is natural to seek a simplification similar to that we have introduced for the non-compact cases. It turns out that there is a simplification but not as effective to reduce the analysis as for the non-compact holonomy cases. This is because the geometry depends on the embedding of $\mathcal{K}_N$ in $\mathcal{P}_L$. Therefore more information is needed to determine the geometry than just the dimension of these spaces.

To give an example where a simplification can be made, consider the two distinct cases that arise in $N = 2$ backgrounds with $\text{hol}(\hat{\nabla}) \subseteq SU(3)$. Inspecting the Killing spinors in [2], it is easy to see that one of these two cases is a special case of $N = 2$ backgrounds with $\text{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$, and the other case is a special case of $N = 2$ backgrounds with $\text{hol}(\hat{\nabla}) \subseteq G_2$. However there are several cases that occur in backgrounds with $\text{hol}(\hat{\nabla}) \subseteq SU(2)$ which do not have such an association. Because of this, we shall describe the geometry of backgrounds for which all parallel spinors are Killing, i.e. $N = L$. The case with $L = 16$ corresponds to the maximally supersymmetric backgrounds and it is known that these are locally isometric to $\mathbb{R}^{9,1}$ [24].

5.2.1 Geometry

The $\hat{\nabla}$-parallel forms which arise as Killing spinor bilinears are

$$e^a \ , \quad \tau \ , \tag{28}$$

where e^a are 1-forms and τ are the fundamental forms of $K \equiv \text{Stab}(\epsilon_1, \ldots \epsilon_L)$, $\text{hol}(\hat{\nabla}) \subseteq K$. The number of parallel 1-forms depends on K and one of them is always time-like. The minimal number of parallel 1-forms are 3,4 and 6 for G_2, $SU(3)$ and $SU(2)$, respectively.

Let e_a denote the dual vector field of e^a, $e^a(\cdot) = g(e_a, \cdot)$. Provided that $dH = 0$, the commutator $[e_a, e_b]$ is again a $\hat{\nabla}$-parallel vector field. The Killing spinor equations in most cases do not put sufficient restrictions on the commutator $[e_a, e_b]$ to express it in terms of the original vector field e_a. So potentially $\mathcal{M}$ may admit more $\hat{\nabla}$-parallel vector fields than those constructed from Killing spinor bilinears. To simplify the analysis that follows, we shall assume that the vector fields constructed form Killing spinor bilinears span a Lie algebra under Lie brackets. Assuming that the action of the vector fields can be integrated to a free group action, $\mathcal{M}$ is a principal bundle, $\mathcal{M} = P(G, B; \pi)$, where the fibre G has Lie algebra that of the vector field $\{e_a\}$ and B is the base space.

$\mathrm{Stab}(\epsilon_1, \ldots, \epsilon_L)$	$1 - \mathrm{forms}$	$\mathfrak{Lie}\, G$
G_2	3	$\mathbb{R}^3$, $\mathfrak{sl}(2, \mathbb{R})$
$SU(3)$	4	$\mathbb{R}^4$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{su}(2) \oplus \mathbb{R}$, $\mathfrak{cw}_4$
$SU(2)$	6	$\mathbb{R}^6$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$, $\mathfrak{cw}_6$

Table 4: In the first column, the compact isotropy groups of spinors are stated. In the second column, the number of 1-form spinor bilinear is given. In the third column, the associated Lorentzian Lie algebras are exhibited. The structure constants of the 6-dimensional Lorentzian Lie algebras of the $SU(2)$ case are self-dual.

Moreover $\mathcal{M}$ is equipped with a principal bundle connection $\lambda^a \equiv e^a$. The Lie algebras of the fibre groups G have been tabulated in table 4.

Using the principal bundle data, the solution of the Killing spinor equations can be expressed as

$$
\begin{aligned}
ds^2 &= \eta_{ab}\, \lambda^a \lambda^b + \pi^* \tilde{g} \\
H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b + \pi^* \tilde{H} \ ,
\end{aligned}
\tag{29}
$$

where

$$
\mathcal{F}^a \equiv d\lambda^a - \frac{1}{2} H^a{}_{bc}\lambda^b \wedge \lambda^c \ ,
\tag{30}
$$

is the curvature of λ and $\tilde{g} = \delta_{ij}\, e^i e^j$ is a metric on B. Apart from the above conditions which are universal, the Killing spinor equations impose additional restrictions on the geometry of spacetime which depend on K. These will be given when we describe each case separately. Observe that H is the sum of the Chern-Simons form of λ and a 3-form $\tilde{H}$ of B. From now one, the forms on B will be denoted with a tilde.

5.2.2 G_2

Let e^a, $a = 0, 1, 2$ and $\varphi = \tilde{\varphi}$ be three 1-forms and the fundamental G_2 form, respectively. In addition to (29), the Killing spinor equations imply

$$
\tilde{H} = -\frac{1}{6}(\tilde{d}\varphi, \star\varphi)\, \varphi + \star\tilde{d}\varphi - \star(\tilde{\theta} \wedge \varphi)
\tag{31}
$$

and

$$
\begin{aligned}
\partial_a \Phi = 0 \ , &\quad \mathcal{F} \in \mathfrak{g}_2 \ , \quad \epsilon^{abc}H_{abc} + H_{ijk}\varphi^{ijk} = 0 \ , \\
\tilde{\theta} = 2d\Phi \ , &\quad \tilde{d}\star\varphi = -\tilde{\theta} \wedge \star\varphi \ ,
\end{aligned}
\tag{32}
$$

where

$$
\tilde{\theta} = -\frac{1}{3} \star (\star\tilde{d}\varphi \wedge \varphi) \ ,
\tag{33}
$$

is the Lee form of φ, and $\tilde{d}$ and $\star$ is the exterior derivative and the Hodge operation on B, respectively. It is clear that the dilaton is invariant under all e_a vector field and so is

a function of B. Moreover $\mathcal{F} \in \mathfrak{g}_2$ implies that the principal bundle connection is a G_2 instanton of B. Another consequence of the same condition is that φ is invariant and since $i_a\varphi = 0$, it is the pull-back of a form on B. In fact the 7-dimensional manifold B admits a G_2 structure. All the conditions in (32) arise from the dilatino Killing spinor equation apart from the last one. This is required [25] for $(B, \tilde{g}, \tilde{H})$ to admit a compatible metric connection with skew-symmetric torsion, $\hat{\tilde{\nabla}}$, and holonomy contained in G_2.

The expression for $\tilde{H}$ depends on whether G is abelian or not. If G is abelian, then the first term in (31) for $\tilde{H}$ vanishes as can be seen from (32). On the other hand if $G = SL(2, \mathbb{R})$, then the same term becomes proportional to the volume of $SL(2, \mathbb{R})$.

5.2.3 $SU(3)$

Let e^a, $a = 0, 1, 2, 3$, and $\omega = \tilde{\omega}$ and $\chi = \tilde{\chi}$ be four 1-forms, and the Hermitian and (3,0) fundamental forms of $SU(3)$, respectively. In addition to (29), the Killing spinor equations imply that

$$\tilde{H} = -i_{\tilde{I}}\tilde{d}\omega = \star \tilde{d}\omega - \star(\tilde{\theta}_\omega \wedge \omega) \,, \tag{34}$$

and

$$
\begin{aligned}
&\partial_a \Phi = 0 \,, \quad \frac{1}{3!}\epsilon^{abcd}H_{bcd} - \frac{1}{2}\mathcal{F}^a_{ij}\omega^{ij} = 0 \,, \quad (\mathcal{F}^a)^{2,0} = 0 \,, \\
&\tilde{\mathcal{N}}(I) = 0 \,, \quad \tilde{\theta}_\omega = \tilde{\theta}_{\mathrm{Re}\chi} \,, \\
&\partial_i \Phi - \frac{1}{2}(\tilde{\theta}_\omega)_i = 0 \,,
\end{aligned}
\tag{35}
$$

where

$$\tilde{\theta}_\omega = -\star(\star\tilde{d}\tilde{\omega} \wedge \tilde{\omega}) \,, \quad \tilde{\theta}_{\mathrm{Re}\chi} = -\frac{1}{2}\star(\star\tilde{d}\mathrm{Re}\chi \wedge \mathrm{Re}\chi) \,, \tag{36}$$

are the Lee forms of ω and χ, respectively. The conditions have similarities with those of both the G_2 and $SU(4) \ltimes \mathbb{R}^8$ cases. The dilaton Φ is a function of B.

The geometry of B depends on whether G is abelian or non-abelian. If G is abelian, then $(B, \tilde{g}, \tilde{H})$ is a complex manifold that admits a compatible metric connection, $\hat{\tilde{\nabla}}$, with skew-symmetric torsion and with holonomy contained in $SU(3)$. This follows from

$$\mathcal{F} \in \mathfrak{su}(3) \,, \tag{37}$$

which in turn implies that ω and χ are invariant under the action of all vector field e_a, and the equality of Lee forms in (35). Moreover λ is a Donaldson type of connection. Since $2d\Phi = \tilde{\theta}_\omega$, B is conformally balanced[5].

Next suppose that G is non-abelian and so is locally either $\mathbb{R} \times SU(2)$ or $SL(2, \mathbb{R}) \times U(1)$. In such a case,

$$\mathcal{F} \in \mathfrak{su}(3) \oplus \mathbb{R} \,, \tag{38}$$

---

[5] It is known that the smooth compact $2n$-dimensional conformally balanced manifolds B with $\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq SU(n)$ and with $\tilde{d}\tilde{H} = 0$ [19] are Calabi-Yau with $\tilde{H} = 0$. However, there are non-compact smooth examples.

and χ is not invariant under the $\mathbb{R}$ and $U(1)$ group actions, respectively. Instead it is invariant up to a $U(1)$ rotation. As a result, the canonical bundle of B is twisted and so B has not an $SU(3)$ structure but rather a $U(3)$ one. So in this case, $(B, \tilde{g}, \tilde{H})$ is a Hermitian manifold with a compatible connection, $\hat{\tilde{\nabla}}$, with skew-symmetric torsion and with holonomy contained in $U(3)$.

5.2.4 $SU(2)$

Let e^a, $a = 0, \ldots, 5$, and $\omega_r = \tilde{\omega}_r$, $r = 1, 2, 3$, be six 1-forms and the three Hermitian fundamental forms of $SU(2)$, respectively, where the endomorphism $I_r I_s = -\delta_{rs} \mathbf{1}_{4 \times 4} + \epsilon_{rst} I_t$. In addition to the conditions (29), the Killing spinor equations imply that

$$\tilde{H} = -i_{I_1} \tilde{d}\omega_1 \tag{39}$$

and

$$\partial_a \Phi = 0 \ , \qquad H_{a_1 a_2 a_3} + \frac{1}{3!} \epsilon_{a_1 a_2 a_3}{}^{b_1 b_2 b_3} H_{b_1 b_2 b_3} = 0 \ ,$$
$$i_{I_r} \tilde{d}\omega_s = i_{I_s} \tilde{d}\omega_r \ , \quad r \neq s \ , \quad \mathcal{F}^a \in \mathfrak{su}(2) \ , \quad 2\partial_i \Phi - (\tilde{\theta}_{\omega_1})_i = 0 \ , \tag{40}$$

where $\tilde{\theta}_{\omega_1}$ is the Lee form of ω_1 as in (24). Again Φ is a function of the base space B. In this case, the Lie algebra of the fibre G is self-dual, i.e. the structure constants satisfy the self-duality condition. In addition, the principal bundle connection λ is an anti-self-dual instanton. It turns out that the conditions (40) imply that the base space B is *conformally hyper-Kähler*.

6 All half supersymmetric solutions

These are the solutions of both the Killing spinor and field equations that admit 8 Killing spinors. There are three classes of such backgrounds which have been classified in [4]. One class is that of $N = L = 8$ backgrounds with $\text{Stab}(\epsilon_1, \ldots, \epsilon_L) = SU(2)$ investigated in 5.2.4. The other class is that of $N = L = 8$ backgrounds with $\text{Stab}(\epsilon_1, \ldots, \epsilon_L) = \mathbb{R}^8$ examined in 5.1.5. The third class is that of $(L, N) = (16, 8)$ backgrounds associated with $\text{Stab}(\epsilon_1, \ldots, \epsilon_{16}) = \{1\}$. It turns out that the third case is a special case of the other two. So we have only two possibilities to investigate.

6.1 $SU(2)$

It has been demonstrated that the spacetime in this case is a principal bundle $\mathcal{M} = P(B, G; \pi)$ over a conformally hyper-Kähler manifold B, equipped with a anti-self-dual connection λ and fibre group G with a self-dual Lorentzian Lie algebra. Using these data, one can write

$$ds^2 = \eta_{ab} \lambda^a \lambda^b + h \, ds_{\text{hk}}^2 \ , \quad e^{2\Phi} = h \ ,$$
$$H = \frac{1}{3} \eta_{ab} \lambda^a \wedge d\lambda^b + \frac{2}{3} \eta_{ab} \lambda^a \wedge \mathcal{F}^b - \star_{\text{hk}} \tilde{d}h \ , \tag{41}$$

where h is a function of B and ds^2_{hk} is a 4-dimensional hyper-Kähler metric.

To find explicit examples one has to specify a 4-dimensional hyper-Kähler manifold, a anti-self dual instanton connection over it with gauge group G and to determine the function h. The latter is found by exploring the Bianchi identity (3) of H, i.e. $dH = 0$, where we have neglected the anomaly term which is proportional to α'. In particular,

$$dH \equiv d\pi^* \tilde{H} + \eta_{ab} \mathcal{F}^a \wedge \mathcal{F}^b = 0 \ . \tag{42}$$

One can include higher order corrections α' corrections and the complete analysis has been done [4]. This in turn can be written as

$$-\nabla^2_{\text{hk}} h - \frac{1}{2}\eta_{ab}\, \mathcal{F}^a_{ij}\, \mathcal{F}^{bij} = 0 \ , \tag{43}$$

One class of solutions is given by taking λ to be a trivial connection. In such a case the spacetime is $\mathcal{M} = G \times B$ and

$$ds^2 = ds^2(G) + ds^2(B) \ , \quad H = \frac{1}{6}H_{abc}\lambda^a \wedge \lambda^b \wedge \lambda^c \ , \quad e^{2\Phi} = \text{const} \ , \tag{44}$$

i.e. H is determined in terms of the structure constants of G.

An example of a solution with non-trivial connection λ can be constructed by taking $B = \mathbb{R}^4$ and $G = SL(2,\mathbb{R}) \times SU(2)$. In addition consider an anti-self dual connection λ on $\mathbb{R}^4$ with gauge group $SU(2)$ and with instanton number 1. Since only the $SU(2)$ subgroup of G is gauged, the spacetime is $\mathcal{M} = SL(2,\mathbb{R}) \times X_7$. In particular,

$$\begin{aligned}
ds^2 &= ds^2\big(SL(2,\mathbb{R})\big) + \delta_{pq}\lambda^p\lambda^q + h\, ds(\mathbb{R}^4) \ , \quad e^{2\Phi} = h \ , \quad p,q = 1,2,3 \ , \\
H &= d\text{vol}(SL(2,\mathbb{R}) + \frac{1}{3}\delta_{pq}\lambda^p \wedge d\lambda^q + \frac{2}{3}\delta_{pq}\lambda^p \wedge \mathcal{F}^q - \star_{\text{hk}}\tilde{d}h \ ,
\end{aligned} \tag{45}$$

where

$$h = 1 + 4\frac{|x|^2 + 2\rho^2}{(|x|^2 + \rho^2)^2} \ , \quad x \in \mathbb{R}^4 \ , \tag{46}$$

and where ρ is the size of the instanton. This solution easily generalizes to multi-instanton $SU(2)$ solutions [4]. Thus there is a class of solutions which depends on $8\nu - 3$ parameters, the moduli of $SU(2)$ instantons with instanton number ν.

6.2 $\mathbb{R}^8$

The conditions stated in (27) for this case imply that there is a choice of coordinates (u, v, x^i) such that

$$\begin{aligned}
ds^2 &= 2e^- e^+ + ds^2(\mathbb{R}^8) \ , \quad H = d(e^- \wedge e^+) \ , \\
e^- &= h^{-1}dv \ , \quad e^+ = du + V dv + n_i dx^i \ .
\end{aligned} \tag{47}$$

All components of the metric and H depend on v and x, and $e_+ = \partial_u$ is the null parallel vector field.

The solutions of the Killing spinor equations are determined up to the functions h and V, and the 1-form n. These in turn can be found by solving the field equations (4). If in addition one assumes that h, V and n are v independent, then the field equations imply that

$$\partial_i^2 h = \partial_i^2 V = 0 \ , \quad \partial^i dn_{ij} = 0 \ . \tag{48}$$

So h and V are harmonic functions of $\mathbb{R}^8$ and dn satisfies the Maxwell equations on $\mathbb{R}^8$. The solution is a superposition of fundamental strings [26], pp-waves and null rotations.

Acknowledgements

References

[1] U. Gran, P. Lohrmann and G. Papadopoulos, "The spinorial geometry of supersymmetric heterotic string backgrounds," JHEP **0602** (2006) 063 [arXiv:hep-th/0510176].

[2] U. Gran, G. Papadopoulos, D. Roest and P. Sloane, "Geometry of all supersymmetric type I backgrounds," JHEP **0708** (2007) 074 [arXiv:hep-th/0703143].

[3] U. Gran, G. Papadopoulos and D. Roest, "Supersymmetric heterotic string backgrounds," Phys. Lett. B **656** (2007) 119 [arXiv:0706.4407 [hep-th]].

[4] G. Papadopoulos, "New half supersymmetric solutions of the heterotic string," arXiv:0809.1156 [hep-th].

[5] C. M. Hull and P. K. Townsend, "The Two Loop Beta Function For Sigma Models With Torsion," Phys. Lett. B **191** (1987) 115.

[6] J. Gillard, G. Papadopoulos and D. Tsimpis, "Anomaly, fluxes and (2,0) heterotic-string compactifications," JHEP **0306** (2003) 035 [arXiv:hep-th/0304126].

[7] J. Gillard, U. Gran and G. Papadopoulos, "The spinorial geometry of supersymmetric backgrounds," Class. Quant. Grav. **22** (2005) 1033 [arXiv:hep-th/0410155].

[8] U. Gran, J. Gutowski, G. Papadopoulos and D. Roest, "N = 31 is not IIB," JHEP **0702** (2007) 044 [arXiv:hep-th/0606049].

"N = 31, D = 11," JHEP **0702** (2007) 043 [arXiv:hep-th/0610331].

"IIB solutions with $N > 28$ Killing spinors are maximally supersymmetric," JHEP **0712** (2007) 070 [arXiv:0710.1829 [hep-th]].

[9] F. R. Harvey, "Spinors and Calibrations," Academic Press, London (1990).

[10] McKenzie Y. Wang, "Parallel spinors and parallel forms", Ann. Global Anal Geom. Vol 7, No 1 (1989), 59.

[11] J. Figueroa-O'Farrill, T. Kawano and S. Yamaguchi, "Parallelisable heterotic backgrounds," JHEP **0310**, 012 (2003) [arXiv:hep-th/0308141].

T. Kawano and S. Yamaguchi, "Dilatonic parallelizable NS-NS backgrounds," Phys. Lett. B **568** (2003) 78 [arXiv:hep-th/0306038].

[12] A. Medina and P. Revoy, "Alge'bres de Lie et produit scalaire invarinat", Ann. scient, E'c. Norm. Sup **18** (1985) 553.

[13] B. S. Acharya, J. M. Figueroa-O'Farrill, B. J. Spence and S. Stanciu, "Planes, branes and automorphisms. II: Branes in motion," JHEP **9807** (1998) 005 [arXiv:hep-th/9805176].

J. M. Figueroa-O'Farrill, "Breaking the M-waves," Class. Quant. Grav. **17** (2000) 2925 [arXiv:hep-th/9904124].

[14] S. Ivanov, "Connection with torsion, parallel spinors and geometry of $Spin(7)$ manifolds" Math. Res. Lett. **11** (2004), no. 2-3, 171–186 [math.DG/0111216].

[15] S. J. . Gates, C. M. Hull and M. Rocek, "Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models," Nucl. Phys. B **248** (1984) 157.

[16] P. S. Howe and G. Sierra, "Two-Dimensional Supersymmetric Nonlinear Sigma Models With Torsion," Phys. Lett. B **148** (1984) 451.

[17] A. Strominger, "Superstrings With Torsion," Nucl. Phys. B **274**, 253 (1986).

[18] P. S. Howe and G. Papadopoulos, "Ultraviolet Behavior Of Two-Dimensional Supersymmetric Nonlinear Sigma Models," Nucl. Phys. B **289** (1987) 264.

"Further Remarks On The Geometry Of Two-Dimensional Nonlinear Sigma Models," Class. Quant. Grav. **5** (1988) 1647.

[19] S. Ivanov and G. Papadopoulos, "A no-go theorem for string warped compactifications," Phys. Lett. B **497** (2001) 309 [arXiv:hep-th/0008232].

"Vanishing theorems and string backgrounds," Class. Quant. Grav. **18** (2001) 1089 [arXiv:math.dg/0010038].

[20] G. Papadopoulos and A. A. Tseytlin, "Complex geometry of conifolds and 5-brane wrapped on 2-sphere," Class. Quant. Grav. **18** (2001) 1333 [arXiv:hep-th/0012034].

[21] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, "G-structures and wrapped NS5-branes," Commun. Math. Phys. **247** (2004) 421 [arXiv:hep-th/0205050].

[22] S. Chiossi and S. Salamon, "The intrinsic torsion of $SU(3)$ and G_2 structures," arXiv:math.dg/0202282.

A. Fino, M. Parton and S. Salamon, "Families of strong KT structures in six dimensions", [math.DG/0209259].

[23] G. L. Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis and G. Zoupanos, "Non-Kaehler string backgrounds and their five torsion classes," Nucl. Phys. B **652** (2003) 5 [arXiv:hep-th/0211118].

[24] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Maximally supersymmetric solutions of ten- and eleven-dimensional supergravities," JHEP **0303**, 048 (2003) [arXiv:hep-th/0211089].

[25] T. Friedrich and S. Ivanov, "Parallel spinors and connections with skew-symmetric torsion in string theory" Asian Journal of Mathematics **6** (2002), 303-336 [math.DG/0102142].

"Killing spinor equations in dimension 7 and geometry of integrable G_2-manifolds" J. Geom. Phys. 48 (2003), 1-11 [math.DG/0112201].

[26] A. Dabholkar, G. W. Gibbons, J. A. Harvey and F. Ruiz Ruiz, "Superstrings and solitons," Nucl. Phys. B **340** (1990) 33.